Phyx 320
Modern Physics

Quiz5:

$R \neq 10^{-11} m$

$R = 10^{-10} m$

No lab Wed.

March 29, 2021

Reading: 40.5-40.8

Homework #9 and Reading Reflection Thursday 11:59 pm

Finite Potential Wells

Two types of states:

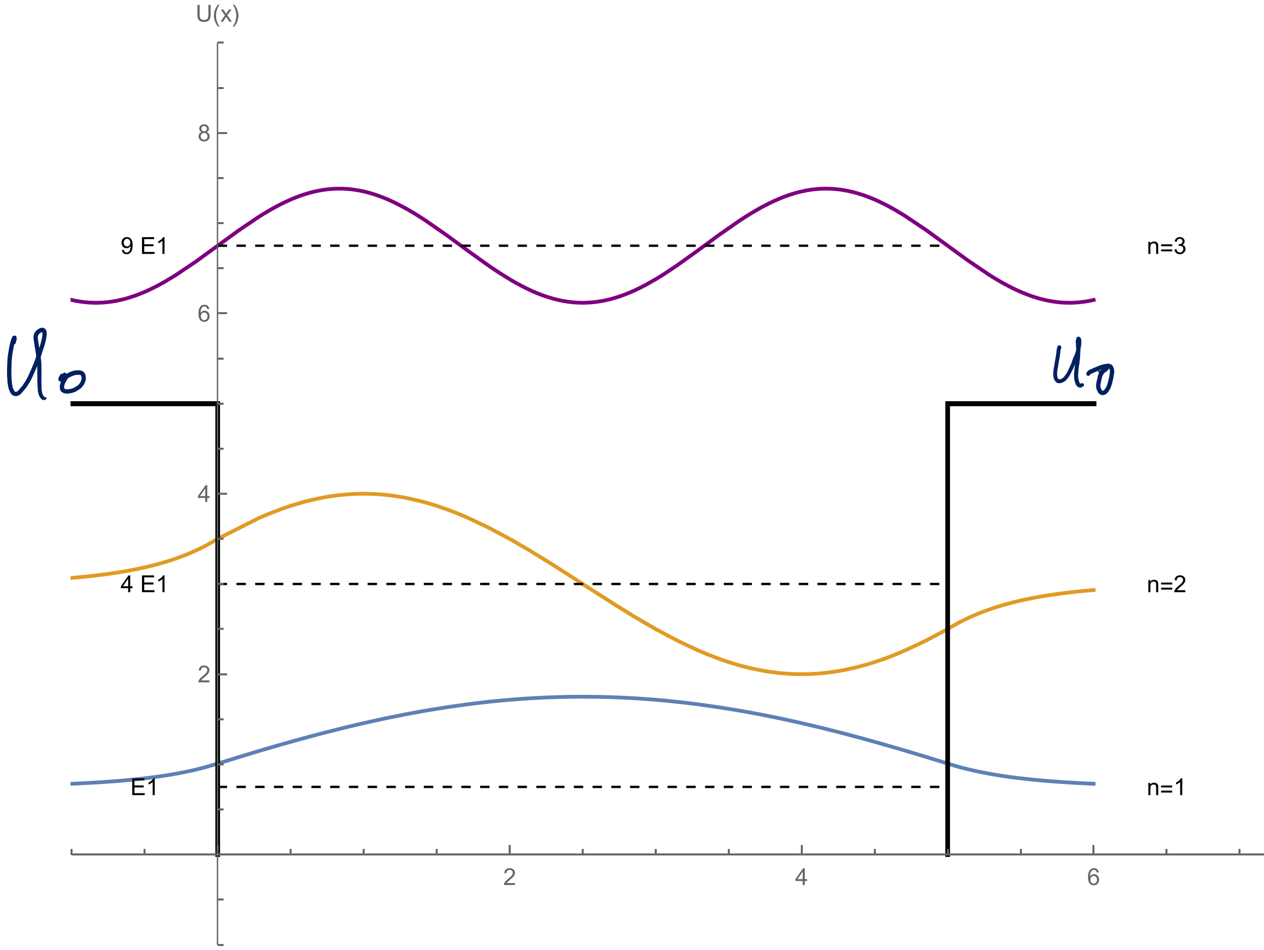
- Bound states ($E < U_0$)
- Free states ($E > U_0$)

In classically forbidden region, wavefunction decays exponentially:

$$\psi(x) = \psi_{edge}\, e^{-\frac{x-L}{\eta}}$$

Wavefunction decays with a characteristic length scale, penetration depth:

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$$



Harmonic Oscillator

Harmonic potential:

$$\mathrm{U}(x) = \frac{1}{2}\kappa\, x^2$$

Classical systems:

- Mass on a spring
- Pendulums
- Acoustic systems
- Circuits

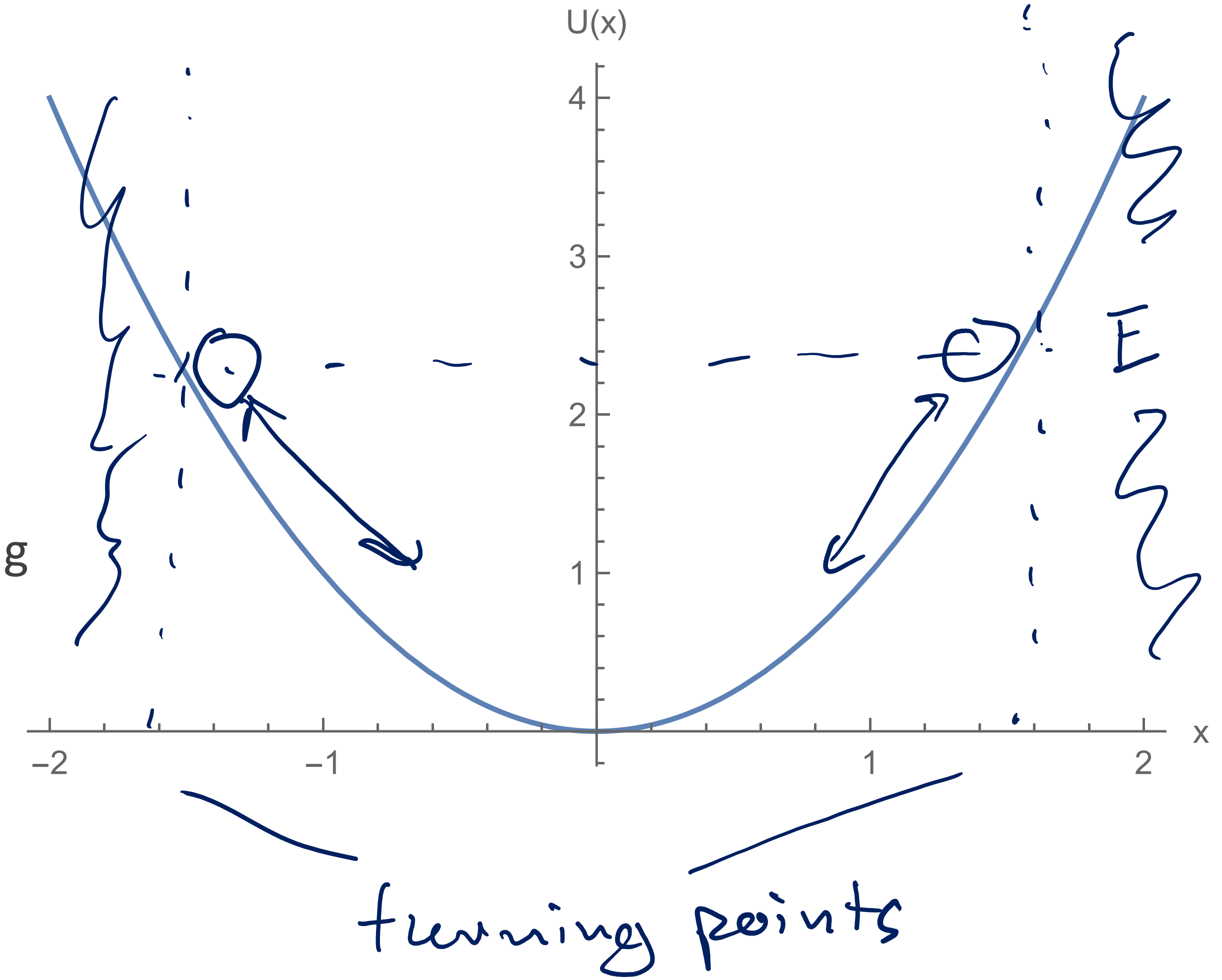
Quantum systems:

- Molecular vibrations
- Particles in optical traps
- Photons
- Sound (phonons)

Classically can imagine the system being a ball rolling up and down the potential (without friction)

Classically forbidden to be in regions where:

$$U(x) > E$$



Harmonic Oscillator

Let's solve this classically:

$$U = \frac{1}{2} k x^2$$

$$F = -\frac{dU}{dx} = -\frac{1}{2} k (2x)$$

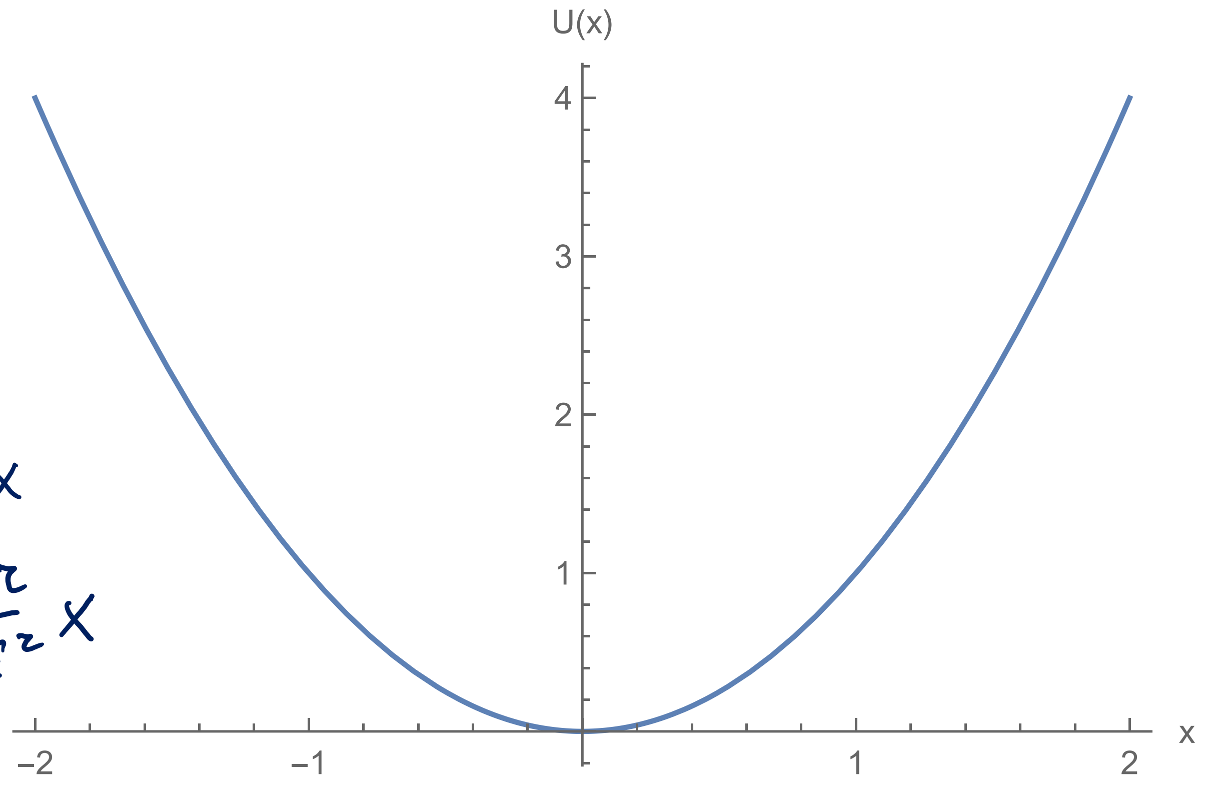
$$= -kx$$

$$F = m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m} x$$

$$\dot{x} = \frac{d}{dt} x$$

$$\ddot{x} = \frac{d^2}{dt^2} x$$



Harmonic Oscillator

Let's solve this classically:

$$x = A\cos(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = A\left(-\omega^2\cos(\omega t + \phi)\right)$$

$\ddot{x}$

$$\ddot{x} = -\frac{k}{m}x$$

$$-\omega^2 A\cos(\omega t+\phi) = -\frac{k}{m}A\cos(\omega t+\phi)$$

$$\omega^2 = \frac{k}{m}$$

$$\boxed{\omega = \sqrt{k/m}}$$

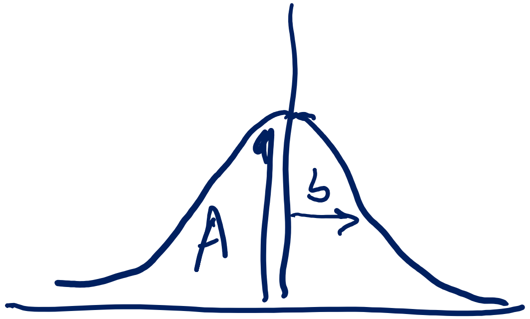
resonant or natural frequency

Harmonic Oscillator

Now quantum mechanically:

$$\psi_1(x) = A_1 e^{-x^2/2b^2}$$

gaussian



$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - U(x)\right)\psi(x)$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi(x)$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left[A_1 e^{-x^2/2b^2}\left(-\frac{2x}{2b^2}\right)\right]$$

$$= A_1 e^{-x^2/2b^2}\left(-\frac{x}{b^2}\right)^2 + A_1 e^{-x^2/2b^2}\left(-1/b^2\right)$$

$$= A_1 e^{-x^2/2b^2}\left(x^2/b^4 - 1/b^2\right)$$

Harmonic Oscillator

Now quantum mechanically:

$$A_1 e^{-x^2/2b^2}\left[\frac{x^2}{b^4} - 1/b^2\right] = -\frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)A_1 e^{-x^2/2b^2}$$

$$\frac{x^2}{b^4} - \frac{1}{b^2} = \frac{km}{\hbar^2}x^2 - \frac{2mE}{\hbar^2}$$

$$\left(\frac{1}{b^4} - \frac{km}{\hbar^2}\right)x^2 + \left(\frac{2mE}{\hbar^2} - \frac{1}{b^2}\right) = 0$$

$$\Downarrow$$

$$\frac{1}{b^4} = \frac{km}{\hbar^2}$$

$$\omega = \sqrt{k/m}$$

$$k = m\omega^2$$

$$b^4 = \frac{\hbar^2}{m^2\omega^2} \Rightarrow \boxed{b = \sqrt{\frac{\hbar}{m\omega}}}$$

Harmonic Oscillator

Now quantum mechanically:

$$b = \sqrt{\frac{\hbar}{m\omega}} \longrightarrow \frac{2mE_1}{\hbar^2} - \frac{1}{b^2} = 0$$

$$\frac{2mE_1}{\hbar^2} = \frac{m\omega}{\hbar}$$

$$E_1 = \frac{1}{2}\hbar\omega$$

Harmonic Oscillator

Higher states:

$$\psi_1(x) = A_1 e^{-\frac{x^2}{2b^2}}$$

Hermite Polynomials

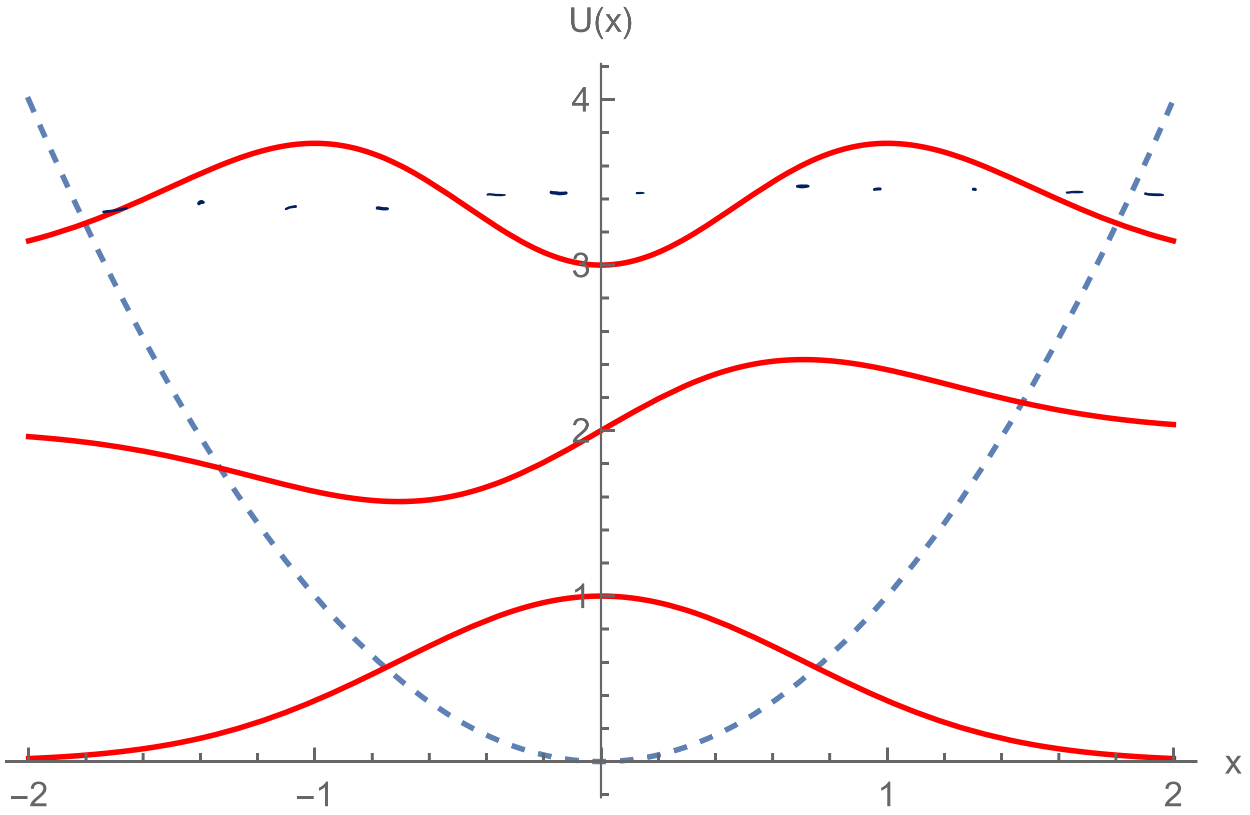
$$\psi_2(x) = A_2 \frac{x}{b} e^{-\frac{x^2}{2b^2}}$$

$$\psi_3(x) = A_3 \left(1 - \frac{2x^2}{b^2}\right) e^{-\frac{x^2}{2b^2}}$$

$$b = \sqrt{\frac{\hbar}{\cancel{2}\omega m}}$$

Energies follow:

$$E_n = \left(n - \frac{1}{2}\right) \hbar\omega$$



Harmonic Oscillator

We can also use Heisenberg's uncertainty principle to derive the ground state energy:

$$E = K + U$$

$$= \frac{p^2}{2m} + \frac{1}{2} k x^2$$

$$E_1 = \frac{(\Delta p)^2}{2m} + \frac{1}{2} k (\Delta x)^2$$

$$= \frac{\hbar^2}{8m} \frac{1}{\Delta x^2} + \frac{1}{2} k \Delta x^2$$

$$\Delta x \Delta p \geq \hbar/2$$

minimum:

$$\Delta x \Delta p = \hbar/2$$

$$x = 0 + \Delta x$$

$$p = 0 + \Delta p$$

$$\Delta p = \frac{\hbar}{2} \frac{1}{\Delta x}$$

Harmonic Oscillator

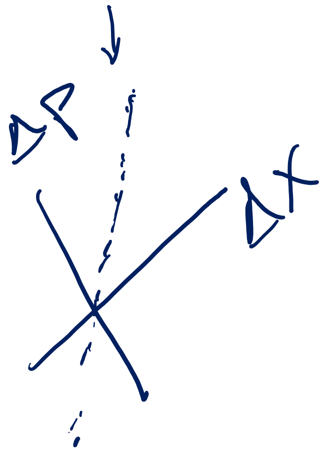
We can also use Heisenberg's uncertainty principle to derive the ground state energy:

$$E_1 = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}k\Delta x^2$$

$\Delta x \Delta p = \hbar/2$

$$\frac{dE_1}{d(\Delta x)} = 0 = -2\frac{\hbar^2}{8m\Delta x^3} + k\Delta x \qquad k = \omega^2 m$$

$$\frac{\hbar^2}{4m} = k\Delta x^4$$



$$\Delta x^4 = \frac{\hbar^2}{4m^2\omega^2}$$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}} \Rightarrow \Delta p = \frac{\hbar}{2}\frac{1}{\Delta x} = \frac{\hbar}{2}\sqrt{\frac{2m\omega}{\hbar}} = \sqrt{\frac{\hbar m\omega}{2}}$$

Harmonic Oscillator

We can also use Heisenberg's uncertainty principle to derive the ground state energy:

$$E_1 = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2}k\Delta x^2 \qquad \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$= \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}k\frac{\hbar}{2m\omega} \qquad k = m\omega^2$$

$$= \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega$$

$$= \frac{1}{2}\hbar\omega$$

Harmonic Oscillator

Ground state energy can be determined by Heisenberg Uncertainty Principle

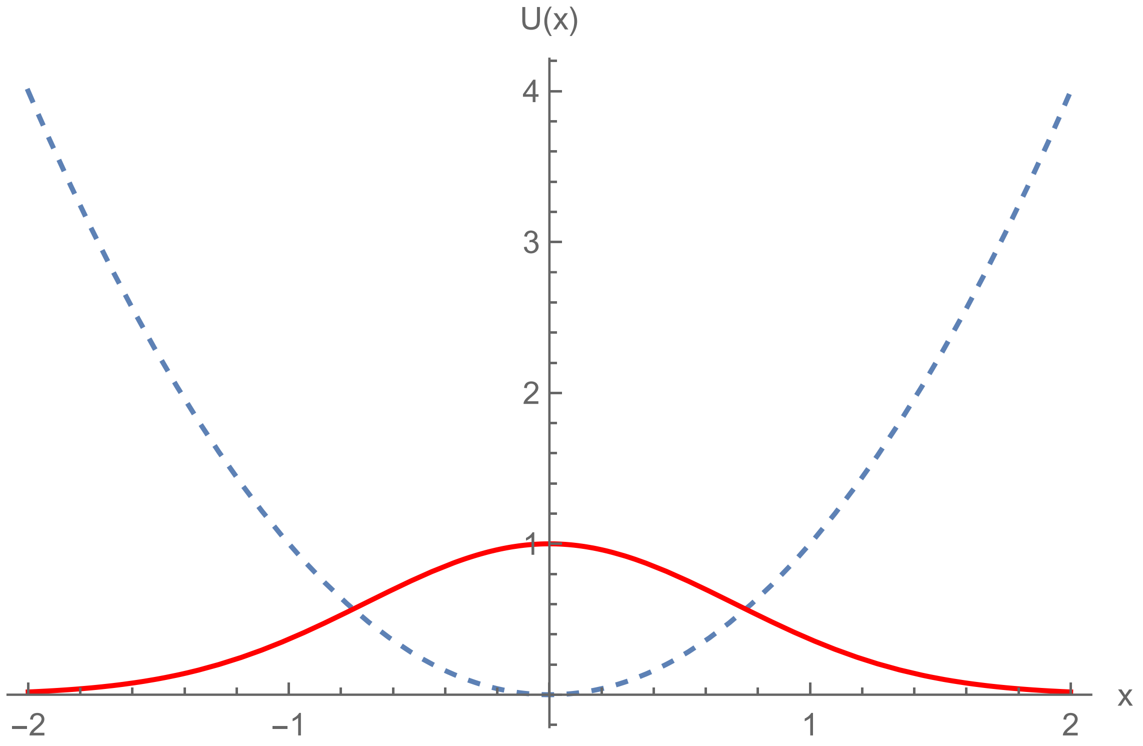
$$E_1 = \frac{1}{2}\hbar\omega$$

Since this is non-zero, a particle in a harmonic trap can never be stationary leading to zero-point motion

Lowest energy level restricted by Heisenberg uncertainty principle

This zero-point energy keeps liquid Helium from freezing at atmospheric pressures, even at absolute zero

True for any particle that is confined to a range of locations



Homework Questions

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